حل المعادلات التفاضلية الخطية الجزئية المتجانسة واللامتجانسة باستخدام تحويلات لابلاس

Solving Homogenous and Non Homogenous Linear partial Differential Equations by Laplace transformations

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المستخلص:

هدفنا في هذا البحث هو حل المعادلات التفاضلية الجزئية الخطية المتجانسة واللامتجانسة ذات المعاملات الثابتة باستخدام تحويلات لابلاس

١. عندما تكون المعادلة خاضعة لبعض الشروط.

٢. بدون استخدام أي شرط.

Abstract

Our aim in this paper is to solve the homogenous and nonhomogenous linear partial differential equations with constant coefficients by using Laplace transformations:-

1. with some initial conditions.

2. Without any conditions.

Introduction

Laplace Transformation [5] is an important transformations for solving linear ordinary and partial differential

equations with constant coefficients which subjected to some initial or boundary conditions.

In [3] Mohammed, A. H was suggested different method from the usual method about solving linear ordinary differential equations by using Laplace transformations (L.T) with constant coefficients and submissive for initial conditions, also in [2] Mohammed, A. H suggested another method for solving linear ordinary differential equations with constant coefficients and without using any conditions by method of Laplace transformations also. In our work, we generalized the above two methods for solving the linear partial differential equations with constant coefficients .

1. with some initial conditions

2. without using any conditions.

In [3] Mohammed, A. H supposed a linear ordinary differential equation of order (n) with constant coefficients and due to certain initial conditions, which general form can be written as follows:-

$$a_0\, y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = f(x)$$

Where f(x) is continuous function whose Laplace transformation can be determined and $y(0), \ldots, y^{(n-1)}(0)$ are defined, after he taking (L.T) to both sides to the above equation he put

$$L(y) = \frac{h(s)}{k(s)}$$

Where h and k are polynomials on (s), such that the degree of (h) is less than the degree of (k) and the polynomial (k) with known prime factors and by taking inverse Laplace transformation $(L^{-1}.T)$ to the last equation, he put the general solution as follows:

$$y = A_0 k_0(x) + \ldots + A_m k_m(x)$$

Such that $k_0,...,k_m$ are function of x (m equal to the degree of k(s)) and that $A_0,...,A_m$ are constants. To find the values of $A_i\ (i=0,...,m)$ he substituted the initial conditions, one of them is y(0) and by taking derivatives of the last equation (m) times he got system of linear equations and by solving this linear system he obtained the values of $A_i\ (i=0,...,m)$ and so by this method he obtained the solution of the required differential equation.

In [2] Mohammed , A. H also, supposed the above (L.O.D.E) but without subjected to any initial conditions he took L.T to both sides, he found

$$L(y)=\frac{k(s)}{(a_0s^n+a_1s^{n-1}+...+a_n)h(s)}$$

Where h(s) is polynomial of (s) represents denominator of L.T of the function f(x) and k(s) is also a polynomial of (s) and its degree smaller than the degree of the product of $(a_0s^n+...+a_n)$ and h(s) and after he took $(L^{-1}.T)$ to the above equation he found that

$$y=A_1g_1(x)+...+A_ng_n(x)+B_1h_1(x)+...+B_rh_r(x)$$

Where $A_i's$ and $B_j's$ are constants and $g_j's$ and $h_j's$ are functions of x. The number of constants $B_i's$ and the number of $h_i's$ are equal to the degree of h(s) which supposed to be (r). He noted that the order of (O.D.E) is (n) , therefore its general solution must contain (n) constants only, but the last solution contains (n+r) constants and to solve this problem, he eliminated ($B_j's$) by obtaining whose values by substituting the last solution on (O.D.E), so he found a solution contains only (n) constants and by this method he got the general solution to the above (O.D.E) without using any initial conditions Mohammed, A. H [2] , [3] noted that the differ between his method and the usual method of (L.T) is that the usual method depends on the numerator, but his methods are independent on numerator.

Definition: The Laplace transform of a function f(x) is denoted by L{f(x)} and is defined as the integral of $e^{-sx} f(x)$ between the limits x=0 and $x=\infty$, i.e

$$L\{f(x)\}=\int_0^\infty e^{-sx}\, f(x)d(x)\equiv F(s)$$

The constant parameter (s) is assumed to be positive and large enough to ensure that the product $e^{-sx} f(x)$ converges to zero as $x\to\infty$, for most common functions f(x) see [4].

Definition: Laplace transformation for a function u(x,t) is defined by the following form

$$L\{u(x,t)\}=\int_0^\infty e^{-st}\, u(x,t)\; dt\equiv v(x,s);s>0$$

From above definition, we get the following

$$L\{u_t\}=sv(x,s)-u(x,0)$$

$$L(u_{tt}\}=s^2v(x,s)-su(x,0)-u_t(x,0)$$

$$L\{u_x\}=Dv(x,s);D=\frac{d}{dx}$$

$$L\{u_{xx}\}=D^2v(x,s);D^2=\frac{d^2}{dx^2}$$

Note: For properties of Laplace transformations and it's inverse, see [1], [4].

The Suggested methods about partial Differential Equations:-

1. The first method with some initial conditions, this method is summarized as follows:-

Suppose we have a P.D.E of order (m+n) with constant coefficients due to certain initial conditions, which general form can be written as follows :

$$\sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij} \frac{\partial^{i+j} u}{\partial t^i \partial x^j} = f(x,t) \quad \ldots (1)$$

Where a_{ij} are constants, f(x,t) is a continuous function whose (L.T) can be determined, and some initial conditions $u(x,0), u_t(x,0), \ldots,$ are defined.

To find the solution of the P.D.E (1) we can take (L.T) to both sides of it and after substituting the conditions, we obtain an O.D.E of order (n) .

The general form of it is

$$a_0 v^{(n)} + a_1 v^{(n-1)} + \ldots + a_n v = g(x,s) \qquad (2)$$

Such that $a_0, a_1, \ldots, a_n$ depend on (a_{ij}) and (s)

The later O.D.E its dependent variable is (v) and the independent variable is (x) , such that v=L {u} .

After this, we write equation (2) by $(D = \frac{d}{dx})$ operator.

Thus we get

$$(a_0 D^n + a_1 D^{n-1} + \ldots + a_n) v = g(x,s) \qquad (3)$$

From equation (3) , we get

$$v = \frac{1}{N(D,s)} . g(x,s) \qquad (4)$$

Where N(D,s) is a function of D and s.

And after solving equation (4) , we get

$$v = \frac{\overline{g}(x,s)}{\overline{N}(s)} \qquad (5)$$

Where $\overline{g}(x,s)$ is a function of (x) and (s) and may be equal to g(x,s) or not, and $\overline{N}(s)$ is a function of (s).

Now, since $u = L^{-1}(v)$, then

$$u = L^{-1}\{\frac{\overline{g}(x,s)}{\overline{N}(s)}\} \qquad (6)$$

Equation (6) represents the solution of the P.D.E (1), which we can be written by the form:

$$u = A_0(x)k_0(t) + A_1(x)k_1(t) + ... + A_m(x)k_m(t) \qquad (7)$$

Such that $k_0, k_1, ..., k_m$ are functions of (t) and that $A_0, A_1, ..., A_m$ are functions of (x) , whose numbers equal to the degree of $\overline{N}(s)$.

The Heat Equation [6]

The heat equation (one dimensional heat equation and without heating source) has the form

$$u_t = \alpha^2 u_{xx}$$

Where u(x,t) represents the temperature at the point (x) and time (t),

$U_t(x,t)$ represents the average of temperature change,

$U_x(x,t)$ represents the slope of temperature curve,

$U_{xt}(x,t)$ represents the convex or concave of temperature curve,

α constant which is called diffusibility and it's equal $\frac{k}{\alpha t}$, where k represents heat conductivity coefficients and α represent density, and all of them are constants.

We shall show the above method by the following examples:-

Example (I) : To solve the irreducible P.D.E (Heat Flow Equation)

$$u_t = u_{xx} \qquad (8)$$

With only one initial condition u(x,0) $= 3\sin 2\pi x$, we take (L.T) to both sides of equation (8) , then

$$-u(x,0) + sv = \frac{d^2v}{dx^2}$$

$$\Rightarrow \frac{d^2v}{dx^2} - sv = -3\sin 2\pi x \qquad (9)$$

From equation (9) , we have

$$v = \frac{1}{D^2 - s}(-3\sin 2\pi x)$$

Thus, $v = \dfrac{3\sin 2\pi x}{s+4\pi^2}$

Since, $u = L^{-1}(v)$

So, $u = 3e^{-4\pi^2 t}\sin 2\pi x$ (represents the solution of (8)).

Example (2) : To solve the non homogenous reducible with homogenous terms P.D.E

$u_x + 4u_t = -8t \qquad \ldots (10)$

With initial condition u(x,0) =0 , we take (L.T) for both sides , we get

$\dfrac{dv}{dx} + 4sv = -\dfrac{8}{s^2} \qquad \ldots (11)$

From equation (11) we get

$$v = \frac{1}{(D+4s)} \cdot -\frac{8}{s^2}$$

$$\Rightarrow v = \frac{-2}{s^3}$$

Since, $u = L^{-1}\{v\}$

So, $u = -t^2$ (represents the solution of (10))

Example (3): To solve the reducible P.D.E (wave Equation) with homogenous terms

$u_{tt} = u_{xx} \qquad \ldots \ (12)$

With only two initial conditions $u(x,0) = \sin \pi x = -u_t(x,0)$ we take (L.T) to both sides of equation (12) , we get

$$\frac{d^2v}{dx^2} - s^2 v = (1-s)\sin \pi x$$

Thus

$$v = \frac{1}{D^2 - s^2}(1-s)\sin \pi x$$

$$= \frac{s-1}{s^2+\pi^2}\sin \pi x$$

Since $u = L^{-1}(v) \quad \Rightarrow u = \sin \pi x(\cos \pi t - \frac{1}{\pi}\sin \pi t)$

Represents the solution of equation (12).

2. Now, we will explain how we can use Laplace transformation to solve L.P.D.E without using any initial and boundary conditions.

We will show this method through out second order linear partial differential equation with constant coefficients which has the general form

$$Au_{xx} + Bu_{xt} + C\, u_{tt} + ju_x + Eu_t + Fu = f(x,t) \qquad (13)$$

Which doesn't obey to any initial and boundary conditions and we suppose that the (L.T) of f(x,t) is known.

To solve equation (13) we take (L.T) to both sides, we get

$$(AD^2 + BsD + Cs^2 + JD + Es + F)v = \frac{h_1(x,s)}{k_1(s)} \qquad (14)$$

Equation (14) is an O.D.E of second order with constant coefficients $A, Bs, J, Cs^2 + Es + F$ where $k_1(s)$ is a polynomial of (s) represents the denominator of L.T of the function f(x,t), and $h_1(x,s)$ is a function of (x) and (s) and it isn't necessary to know the terms of $h_1(x,s)$, we only denoted it by this symbol[2],[3].

To solve equation (14), we have

$$v = \frac{1}{k_1(s)(AD^2 + BsD + JD + Cs^2 + Es + F)} \;.\; h_1(x,s)$$

Now, we'll substitute any constant say (b) instead of D, (because the (L.T) is known for $(e^{ax}, \sin ax, \cos ax, \sinh ax, \cosh ax)$ and its multiplications of these functions, but if the function f(x, t) consists polynomial in (x) we applied the fact

$$\frac{1}{1-D} = (1 + D + D^2 + \ldots) \;,$$

see [1], the constant b may be equal to a or $ai(i=\sqrt{-1})$), that is

$$v=\frac{h_2(x,s)}{k_1(s)(Ab^2+Bsb+Jb+Cs^2+Es+F)}$$

$$=\frac{h_2(x,s)}{k_2(s)}$$

Where $h_2(x,s)$ is a function of (x) and (s) and may be equal to $h_1(x,s)$ or not, $k_2(s)$ is a polynomial of (s)

Since $v=L\{u\}$

$\Rightarrow u=L^{-1}\{\frac{h_2(x,s)}{k_2(s)}\}$, thus we have

$$u=\sum_i B_i(x)\,f_i(t) \qquad (15)$$

Number of i depends on the degree of $k_2(s)$, $B_i(x)$ are functions of (x) and $fi(t)$ are functions of (t), (u) may contain one or more of the functions $B_i(x)$ and $f_i(t)$.

To find the forms of the functions $B_i(x)$, we take derivatives of equation (15) w.r.t (x) and (t) to get u_x, u_{xx}, u_t, u_{xt} and u_{tt}.

We substitute all these derivatives and u in equation (13) and by equality coefficients of both sides we get the values of $B_i(x), \forall i$, and by this method we obtain the solution of the P.D.E (13).

Example 4: - To solve the non- homogenous reducible P.D.E

$$u_{xx}-u_{tt}+u_x+u_t=e^{x+y} \quad \ldots \ (16)$$

By using L.T and without using any conditions, we take L.T to both sides, we get

$$(D^2+D-s^2+s)v=\frac{h(x,s)}{s-1} \ \Rightarrow v=\frac{1}{(s-1)(D^2+D-s^2+s)}\,h(x,s)$$

(h consists e^x)

To find the solution of (v) we substitute (1) instead of the operator(D) (because the cofactor of x for the function e^x equal to one), so, we have

$$v = \frac{\bar{h}(x,s)}{(s-1)(s-2)(s+1)}$$
$$= \frac{A(x)}{s-1} + \frac{B(x)}{s-2} + \frac{C(x)}{s+1}$$

Since $u = L^{-1}\{v\} \Rightarrow \mathrm{u} = \mathrm{A(x)e^{t}} + \mathrm{B(x)e^{2t}} + \mathrm{C(x)e^{-t}}$

Where A, B *and* C are functions of x, and to find the forms of them, we must find u_x, u_{xx}, u_t, u_{tt} and after finding them, we substitute them in P.D.E (16), thus we have

$$A(x) = (c_1 + c_2 e^{-x}) + \frac{1}{2}e^x$$
$$B(x) = c_3 e^{-2x} + c_4 e^x$$
$$C(x) = B(x)$$
$$\Rightarrow u = c_1 e^t + c_2 e^{-x+t} + \frac{1}{2}e^{x+t} + (c_3 e^{-2x} + c_4 e^x)(e^{2t} + e^{-t})$$

Where c_1,c_2 ,c_3 and c_4 are constants.

Example 5 :- To solve the homogenous irreducible P.D.E

$$u_{xx} + u_{tt} + 2u_{xt} + 2u_x + 2u_t + u = 0 \qquad (17)$$

We take (L.T) to both sides, we get

$$(D^2 + 2sD + 2D + s^2 + 2s + 1)v = h(x,s)$$

$$\Rightarrow v = \frac{\bar{h}(x,s)}{a^2 + s^2 + 2s(a+1) + 2a + 1} = \frac{\bar{h}(x,s)}{(s+(a+1))^2}$$ (we substitute (a) instead of operator D)

$$= \frac{A(x)}{s+(a+1)} + \frac{B(x)}{(s+(a+1))^2}$$

Since $u = L^{-1}\{v\}$

$\Rightarrow u = Ae^{-(a+1)t} + Bt\,e^{-(a+1)t}$; A and B are functions of (x).

Now, to find the forms of the functions A and B , we must find u_x, u_{xx}, u_t, u_{tt} and u_{xt} and after finding them, we substitute them in P.D.E (17), thus we find

$A(x) = (c_1 x + c_2)e^{ax} - c_3 x^2\, e^{ax}$

$B(x) = (c_3 x + c_4)e^{ax}$; c_1, c_2, c_3 *and* c_4 *are constants*

$\Rightarrow u = (c_1 x + c_2 - c_3 x^2)e^{ax-(a+1)t} + (c_3 x + c_4)t\, e^{ax-(a+1)t}$

Where c_1, c_2, c_3, c_4 and a are constants.

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